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LETTER TO THE EDITOR

Hopf-type bifurcation in the presence of multiple critical eigenvalues

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Abstract. We show that, assuming a 'weak' form of symmetry in the equations, one can have Hopf-type bifurcation of periodic solutions even in the presence of doubly degenerate critical imaginary eigenvalues.

The problem of finding 'Hopf-type' bifurcations of periodic solutions in the presence of multiple critical eigenvalues is the argument of some recent papers [1, 2]. The existence of bifurcating periodic solutions can be guaranteed by suitable symmetry conditions, and/or by the introduction of a number of 'control parameters' λ larger than one, in contrast with the case of the classical Hopf problem: this point is discussed in detail in the abovementioned references.

In the present letter, we will consider some cases in which a 'weak' form of symmetry is assumed, in such a way that a bifurcation 'à la Hopf' of periodic solutions, with precisely one real control parameter λ, is ensured.

Let $u = u(t) \in R^4$, and consider the non-linear dynamical problem

$$\mathrm{d}u/\mathrm{d}t = f(\lambda; u) \qquad f(\lambda; 0) = 0 \qquad \lambda \in R \tag{1}$$

with the usual smoothness hypothesis on the map $f: R \times R^4 \to R^4$, and assume that the linear part of f (the prime denotes differentiation)

$$L(\lambda) = f'_u(\lambda; 0) \tag{1'}$$

at the critical point $\lambda = \lambda_0$ possesses two imaginary eigenvalues $\sigma = \sigma(\lambda_0) = \pm \mathrm{i}\omega_0$ with double algebraic and geometrical multiplicity. Assume also that $L(\lambda_0)$ is diagonalisable. Then we can write it in the form

$$L(\lambda_0) = \omega_0 K \qquad K = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{2}$$

Introducing the usual scaling in the time variable

$$t \to \tau = \omega t$$

in order that solutions $u = u(\tau)$ are 2π-periodic in the variable τ, (1) can be written

$$\omega\, \mathrm{d}u/\mathrm{d}\tau = (\omega_0 K + L_1(\lambda))u + N(\lambda; u) \tag{3}$$

where $N(\lambda; u)$ is the non-linear higher-order part of f.

Our result is then the following.

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Theorem. Given the problem (1)-(3), assume that there exists a 4×4 real matrix $\hat{S}$, possibly depending on λ, such that

$$\hat{S}^2 = I \qquad (\hat{S} \neq \pm I, \hat{S} = \hat{S}(\lambda)) \tag{4}$$

(I is the identity in R^4) and the following 'symmetry condition':

$$f(\lambda; \hat{S}u) = \hat{S}f(\lambda; u) \qquad \forall \lambda, u \tag{5}$$

is satisfied. Assume also a standard transversality condition $(\mathrm{d}\,\mathrm{Re}\,\sigma/\mathrm{d}\lambda) \neq 0$ at the critical point $\lambda = \lambda_0$, where $\sigma(\lambda)$ is the critical branch of eigenvalues. Then a periodic solution of (3) bifurcates at $\lambda = \lambda_0$.

We sketch the proof only in the special case in which $\hat{S}$ has the form

$$\hat{S} = \begin{pmatrix} 0 & S^{-1} \\ S & 0 \end{pmatrix} \tag{6}$$

where $S = S(\lambda)$ is a 2×2 real non-singular matrix: in this particular case in fact the argument is more suggestive and simple.

Let us introduce the notations

$$\begin{aligned} &u \equiv (u_1, u_2, u_3, u_4) \in R^4 \\ &x \equiv (u_1, u_2) \in R^2 \qquad y \equiv (u_3, u_4) \in R^2 \\ &f \equiv (f_1, f_2, f_3, f_4) \qquad X \equiv (f_1, f_2) \qquad Y \equiv (f_3, f_4). \end{aligned} \tag{7}$$

The point is now to see that equations (3) possess a Hopf-type bifurcation once the vectors y are chosen to be dependent on the vectors x according to the following coupling rule:

$$y = Sx. \tag{8}$$

In fact, from (5), we obtain the identity

$$Y(\lambda; x, Sx) = SX(\lambda; x, Sx) \tag{8'}$$

and equations (3) become

$$\begin{aligned} &\omega\, \mathrm{d}x/\mathrm{d}\tau = X(\lambda; x, Sx) \\ &\omega\, \mathrm{d}y/\mathrm{d}\tau = \omega S\, \mathrm{d}x/\mathrm{d}\tau = Y(\lambda; x, Sx) = SX(\lambda; x, Sx) \end{aligned} \tag{8''}$$

which shows that the problem is reduced to a two-dimensional problem (the second set of equations is equivalent to the first one). Therefore, standard Hopf techniques can be applied, obtaining a periodic solution of the form

$$u = u(\tau) = \begin{pmatrix} x(\tau) \\ Sx(\tau) \end{pmatrix}. \tag{9}$$

In more detail, as a consequence of assumption (5) and precisely from the commutativity with K, S necessarily has the form

$$S = \rho \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \tag{10}$$

for some real $\rho = \rho(\lambda) > 0$ and $\varphi = \varphi(\lambda)$. This implies that the bifurcating solution can be written in the form

$$\begin{aligned} u &= u(\tau) = r\hat{v}(\tau) + O(r^2) && \tau = \omega t \\ \lambda &= \lambda(r) \quad \text{with} && \lambda(0) = \lambda_0 \\ \omega &= \omega(r) \quad \text{with} && \omega(0) = \omega_0 \end{aligned} \tag{11}$$

where the first-order term $\hat{v}(\tau)$ is

$$\hat{v}(\tau) = \begin{pmatrix} \cos \omega t \\ -\sin \omega t \\ \rho(\lambda) \cos(\omega t + \varphi(\lambda)) \\ -\rho(\lambda) \sin(\omega t + \varphi(\lambda)) \end{pmatrix}. \tag{11'}$$

It can be useful—in order to keep as far as possible all these results in a unified context—to rephrase the main point of this letter in the language of [1]. In fact, our matrix $\hat{S}$ generates an action of the group Z_2, and f commutes with this (5). Under this action, on the other hand, Z_2 itself is an isotropy subgroup, and the fixed-point subspace for this group turns out to be two-dimensional: if S has the form (6), this subspace is the set (see (8)) of the vectors $\{(x, Sx)\}$. As in [1], the dynamics leaves this two-dimensional space invariant and there a standard Hopf bifurcation is produced. All this clearly shows, not only the role of symmetry in our result, but also the relationship with the general pattern described in [1].

Just to give an explicit example where all the above assumptions are satisfied, consider the problem

$$\omega\, du/d\tau = [a_0(\lambda) + (a_1(\lambda) + \omega_0)K]u + (4|x|^2 + |y|^2)(a_3(\lambda)u + a_4(\lambda)u^*)$$

where the functions $a_i(\lambda)$ are arbitrary, apart from the requirements

$$a_0(\lambda_0) = a_1(\lambda_0) = 0 \qquad da_0(\lambda_0)/d\lambda \neq 0$$

$|\ |$ is the R^2 norm and u^* is the vector linearly depending on u defined by

$$u^* \equiv (u_3 + u_4, -u_3 + u_4, 4(u_1 - u_2), 4(u_1 + u_2)).$$

With this choice, the matrix S has the form (10) with $\rho = 2$ and $\varphi = \pi/4$.

To conclude, let us mention two elementary particular cases which are naturally included in the above scheme. First, if one has in (6) $S = S^{-1} = I$ (the identity in R^2), then assumption (5) corresponds to a problem which is symmetric with respect to the exchange $x \leftrightarrow y$: then the bifurcating solution clearly has the form $x = y$, i.e. $u_1(\tau) = u_3(\tau)$, $u_2(\tau) = u_4(\tau)$. A second simple instance is obtained if $\hat{S}$ in (5) is

$$\hat{S} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

(which of course does not fall in the particular case (6), and therefore no coupling of the form (8) between the x and y spaces is now present). Condition (5) means in this case that the function $X(x, y)$ is even and $Y(x, y)$ odd with respect to the variable y going into $-y$; the solution we find is then $y(\tau) = 0$, or $u_3(\tau) = u_4(\tau) = 0$.

References

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